

EXTENSION OF LYAPUNOV'S CONVEXITY THEOREM TO SUBRANGES

PENG DAI AND EUGENE A. FEINBERG

ABSTRACT. Consider a measurable space with a finite vector measure. This measure defines a mapping of the σ -field into a Euclidean space. According to Lyapunov's convexity theorem, the range of this mapping is compact and, if the measure is atomless, this range is convex. Similar ranges are also defined for measurable subsets of the space. We show that the union of the ranges of all subsets having the same given vector measure is also compact and, if the measure is atomless, it is convex. We further provide a geometrically constructed convex compactum in the Euclidean space that contains this union. The equality of these two sets, that holds for two-dimensional measures, can be violated in higher dimensions.

1. INTRODUCTION

Let (X, \mathcal{F}) be a measurable space and $\mu = (\mu_1, \dots, \mu_m)$, $m = 1, 2, \dots$, be a finite vector measure on it. For each $Y \in \mathcal{F}$ consider the range $R_\mu(Y) = \{\mu(Z) : Z \in \mathcal{F}, Z \subset Y\} \subset \mathbb{R}^m$ of the vector measures of all its measurable subsets Z . Lyapunov's convexity theorem [9] states that the range $R_\mu(X)$ is compact and furthermore, if μ is atomless, this range is convex. Of course, this is also true for any $Y \in \mathcal{F}$. We recall that a measure ν is called atomless if for each $Z \in \mathcal{F}$ such that $\nu(Z) > 0$, there exists $Z' \in \mathcal{F}$ such that $Z' \subset Z$ and $0 < \nu(Z') < \nu(Z)$. A vector measure $\mu = (\mu_1, \dots, \mu_m)$, is called atomless if each measure μ_i , $i = 1 \dots m$, is atomless. For a review of Lyapunov's convexity theorem and its applications see [10].

Let $\mathcal{S}_\mu^p(X)$ be the set of all measurable subsets of X with the vector measure $p \in R_\mu(X)$,

$$\mathcal{S}_\mu^p(X) = \{Y \in \mathcal{F} : \mu(Y) = p\}.$$

Of course, $\mathcal{S}_\mu^p(X) = \emptyset$, if $p \notin R_\mu(X)$. For $p \in \mathbb{R}^m$ consider the union of the ranges of all subsets of X with the vector measure p ,

$$R_\mu^p(X) = \bigcup_{Y \in \mathcal{S}_\mu^p(X)} R_\mu(Y).$$

In particular, $R_\mu^p(X) = \emptyset$, if $p \notin R_\mu(X)$, and $R_\mu^{\mu(X)}(X) = R_\mu(X)$.

Since the relation $Y_1 = Y_2$ (μ -everywhere) is an equivalence relation on \mathcal{F} , it partitions any subset of \mathcal{F} into equivalence classes. For an atomless μ , Lyapunov [9, Theorem III] proved that: (i) $\mathcal{S}_\mu^p(X)$ consists of one equivalence class if and

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only if p is an extreme point of $R_\mu(X)$, and (ii) if $p \in R_\mu(X)$ is not an extreme point of $R_\mu(X)$, then the set of equivalence classes in $\mathcal{S}_\mu^p(X)$ has cardinality of the continuum.

In general, a union of an infinite number of compact convex sets may be neither closed nor convex. As follows from Dai and Feinberg [1], the set $R_\mu^p(X)$ is a convex compactum, if $m = 2$ and μ is atomless. This fact follows from stronger results that hold for $m = 2$. For $m = 2$ and atomless μ , Dai and Feinberg [1, Theorem 2.3] showed that there exists a set $Z^* \in \mathcal{S}_\mu^p(X)$, called a maximal subset, such that

$$(1.1) \quad R_\mu(Z^*) = R_\mu^p(X),$$

and, in addition, the following equality holds

$$(1.2) \quad R_\mu^p(X) = Q_\mu^p(X),$$

where $Q_\mu^p(X)$ is the intersection of $R_\mu(X)$ with its shift by a vector $-(\mu(X) - p)$,

$$(1.3) \quad Q_\mu^p(X) = (R_\mu(X) - \{\mu(X) - p\}) \cap R_\mu(X),$$

with $S_1 - S_2 = \{q - r : q \in S_1, r \in S_2\}$ for $S_1, S_2 \subset \mathbb{R}^m$. In particular, $R_\mu(X) - \{r\}$ is a parallel shift of $R_\mu(X)$ by $-r \in \mathbb{R}^m$. Examples 3.3 and 3.4 below demonstrate that equalities (1.1) and (1.2) may not hold when μ is not atomless even for $m = 1$.

Each of equalities (1.1) and (1.2) implies that $R_\mu^p(X)$ is convex and compact. However, [1, Example 4.2] demonstrates that a maximal set Z^* may not exist for an atomless vector measure μ when $m > 2$.

In this paper, we prove (Theorem 1) that for any natural number m the set $R_\mu^p(X)$ is compact and, if μ is atomless, this set is convex. This is a generalization of Lyapunov's convexity theorem, which is a particular case of this statement for $p = \mu(X)$. We also prove that $R_\mu^p(X) \subset Q_\mu^p(X)$ (Theorem 2). Example 3.1 demonstrates that it is possible that equality (1.2) may not hold when $m > 2$ and μ is atomless.

Lyapunov's convexity theorem is relevant to purification of transition probabilities discovered by Dvoretzky, Wald, and Wolfowitz [2, 3]. Let (A, \mathcal{A}) be a measurable space and π be a transition probability from (X, \mathcal{F}) to A ; that is, $\pi(B|x)$ is a measurable function on (X, \mathcal{F}) for any $B \in \mathcal{A}$ and $\pi(\cdot|x)$ is a probability measure on (A, \mathcal{A}) for any $x \in X$. According to Dvoretzky, Wald, and Wolfowitz [2, 3], two transition probabilities π_1 and π_2 are called strongly equivalent if

$$(1.4) \quad \int_X \pi_1(B|x) \mu_i(dx) = \int_X \pi_2(B|x) \mu_i(dx), \quad i = 1, \dots, m, \quad B \in \mathcal{A}.$$

A transition probability π is called pure if each measure $\pi(\cdot|x)$ is concentrated at one point. A pure transition probability π is defined by a measurable mapping $\varphi : X \rightarrow A$ such that $\pi(B|x) = I\{\varphi(x) \in B\}$ for all $B \in \mathcal{A}$. According to the contemporary terminology, a transition probability can be purified if it is strongly equivalent to a pure transition probability.

For a finite set A , Dvoretzky, Wald, and Wolfowitz [2, 3] proved that any transition probability can be purified, if the measure μ is atomless. Edwards [5, Theorem 4.5] generalized this result to the case of a countable set A . Khan and Rath [7, Theorem 2] gave another proof of this generalization. Loeb and Sun [8, Example 2.7] constructed an elegant example when a transition probability cannot be purified for $m = 2$, $X = [0, 1]$, $A = [-1, 1]$, and atomless μ . However, purification holds for a countable set of atomless, finite, signed Loeb measures, when A is a complete separable metric space [8, Corollary 2.6]. Podczeck [11] proved that purification holds for a countable set of finite signed measures μ_k absolutely continuous with

respect to a measure μ , when (X, \mathcal{F}, μ) is a super-atomless probability space and A is a compact metric space.

We also mention that for a finite set A , atomless measure μ , and measurable nonnegative functions $f_i, i = 1, \dots, m$, on $X \times A$, Dvoretzky, Wald, and Wolfowitz [3, 4] proved that for any transition probability there exists an equivalent pure transition probability. Feinberg and Piunovskiy [6] proved that this is true for standard Borel spaces X and A . We recall that two transition probabilities π_1 and π_2 are called equivalent [3, 4] if

$$\int_X \int_A f_i(x, a) \pi_1(da|x) \mu_i(dx) = \int_X \int_A f_i(x, a) \pi_2(da|x) \mu_i(dx), \quad i = 1, \dots, m.$$

For a countable set A and atomless μ , define vectors

$$(1.5) \quad p^a = \int_X \pi(a|x) \mu(dx), \quad a \in A.$$

Since purification holds for an atomless μ and a countable A [5], vectors $p^a, a \in A$, can be presented as in (1.5), if and only if there exists a partition $\{X^a : X^a \in \mathcal{F}, a \in A\}$ of the set X such that $\mu(X^a) = p^a$ for all $a \in A$. In fact, in this form the purification theorem was presented by Dvoretzky, Wald, and Wolfowitz [3, 4] for a finite set A .

For $m = 2$, an atomless μ , and a countable A , Dai and Feinberg [1, Theorem 2.5], provided a necessary and sufficient condition that for a set of m -dimensional vectors $\{p^a : a \in A\}$ there exists the above described partition. This condition is that:

$$(1.6) \quad (i) \sum_{a \in A} p^a = \mu(X), \text{ and } (ii) \sum_{a \in B} p^a \in R_\mu(X) \text{ for any finite subset } B \subset A.$$

Obviously, (1.6) is a necessary condition for any natural number m . Example 3.2 below implies that this condition is not sufficient for an atomless μ , when $m > 2$ and A consists of more than two points.

2. MAIN RESULTS

Theorem 2.1. *For any vector $p \in R_\mu(X)$, the set $R_\mu^p(X)$ is compact and, in addition, if the vector measure μ is atomless, this set is convex.*

Proof. We say that a partition is measurable, if all its elements are measurable sets. Consider the set

$$V_{\mu,3}(X) = \{(\mu(S_1), \mu(S_2), \mu(S_3)) : \{S_1, S_2, S_3\} \text{ is a measurable partition of } X\}.$$

According to Dvoretzky, Wald, and Wolfowitz [4, Theorems 1 and 4], $V_{\mu,3}(X)$ is compact and, if μ is atomless, this set is convex. Now let

$$W_\mu^p(X) = \{(s_1, s_2, s_3) : (s_1, s_2, s_3) \in V_{\mu,3}(X), s_3 = \mu(X) - p, s_1 + s_2 = p\}.$$

This set is compact and, if μ is atomless, it is convex. This is true, because $W_\mu^p(X)$ is an intersection of $V_{\mu,3}(X)$ and two planes in \mathbb{R}^{3m} . These planes are defined by the equations $s_3 = \mu(X) - p$ and $s_1 + s_2 = p$ respectively. We further define

$$S_\mu^p(X) = \{s_1 : (s_1, s_2, s_3) \in W_\mu^p(X)\}.$$

Since $S_\mu^p(X)$ is a projection of $W_\mu^p(X)$, the set $S_\mu^p(X)$ is compact and, if μ is atomless, it is convex.

The last step of the proof is to show that $S_\mu^p(X) = R_\mu^p(X)$ by establishing that (i) $S_\mu^p(X) \subset R_\mu^p(X)$, and (ii) $S_\mu^p(X) \supset R_\mu^p(X)$. Indeed, for (i), for any $s_1 \in S_\mu^p(X)$, there exists $(s_1, s_2, s_3) \in W_\mu^p(X)$ or equivalently there exists a measurable partition $\{S_1, S_2, S_3\}$ of X such that $\mu(S_3) = \mu(X) - p$ and $\mu(S_1) + \mu(S_2) = p$. Let $Z = S_1 \cup S_2$. Then $\mu(Z) = p$, $s_1 \in R_\mu(Z)$, and thus $s_1 \in R_\mu^p(X)$. For (ii), for any $s_1 \in R_\mu^p(X)$, there exists a set $Z \in \mathcal{F}$, such that $\mu(Z) = p$ and $s_1 \in R_\mu(Z)$, which further implies that there exists a measurable subset S_1 of Z such that $\mu(S_1) = s_1$. Let $S_2 = Z \setminus S_1$ and $S_3 = X \setminus Z$. Then $\mu(S_1) + \mu(S_2) = p$ and $\mu(S_3) = \mu(X) - p$, which further implies that $(s_1, \mu(S_2), \mu(S_3)) \in W_\mu^p(X)$. Thus $s_1 \in S_\mu^p(X)$. \square

Theorem 2.2. $R_\mu^p(X) \subset Q_\mu^p(X)$ for any vector $p \in R_\mu(X)$.

Recall that $R_\mu^p(X) = Q_\mu^p(X)$ when $m = 2$ and μ is atomless; see (1.2). The proof of Theorem 2.2 uses the following lemma.

Lemma 2.3. ([1, Lemma 3.3]) *For any vector $p \in R(X)$, each of the sets $R_\mu^p(X)$ and $Q_\mu^p(X)$ is centrally symmetric with the center $\frac{1}{2}p$.*

Though it is assumed in [1] that the measure μ is atomless, this assumption is not used in the proofs of Lemmas 3.1-3.3 in [1].

Proof of Theorem 2.2. Let $q \in R_\mu^p(X)$. Since $R_\mu^p(X) \subset R_\mu(X)$, then $q \in R_\mu(X)$. Furthermore, in view of Lemma 2.3, $p - q \in R_\mu^p(X)$. Therefore, $p - q \in R_\mu(X)$. Since $R_\mu(X)$ is centrally symmetric with the center $\frac{1}{2}\mu(X)$, then $R_\mu(X) = \{\mu(X)\} - R_\mu(X)$ and $p - q \in R_\mu(X) = \{\mu(X)\} - R_\mu(X)$. Therefore, $q \in R_\mu(X) - \{\mu(X) - p\}$. As follows from the definition of $Q_\mu^p(X)$ in (1.3), $q \in Q_\mu^p(X)$. \square

3. COUNTEREXAMPLES

The first counterexample shows that equality (1.2) may not hold when μ is atomless and $m > 2$. In particular, the inclusion in Theorem 2.2 cannot be substituted with the equality.

Example 3.1. Consider the measure space (X, \mathcal{B}, μ) , where $X = [0, 6]$, \mathcal{B} is the Borel σ -field on X , and $\mu(dx) = (\mu_1, \mu_2, \mu_3)(dx) = (f_1(x), f_2(x), f_3(x))dx$, where

$$f_1(x) = \begin{cases} 30 & x \in [0, 1), \\ 40 & x \in [1, 2), \\ 10 & x \in [2, 4), \\ 15 & x \in [4, 5), \\ 5 & x \in [5, 6]; \end{cases} \quad f_2(x) = \begin{cases} 40 & x \in [0, 1), \\ 10 & x \in [1, 2), \\ 20 & x \in [2, 4), \\ 10 & x \in [4, 5), \\ 30 & x \in [5, 6]; \end{cases} \quad f_3(x) = \begin{cases} 10 & x \in [0, 1), \\ 20 & x \in [1, 3), \\ 30 & x \in [3, 4), \\ 20 & x \in [4, 5), \\ 25 & x \in [5, 6]. \end{cases}$$

These density functions are plotted in Fig. 3.1. Note that $\mu(X) = (110, 130, 125)$ and

$$(3.1) \quad R_\mu(X) = \left\{ \sum_{i=1}^6 \alpha_i p^i : \alpha_i \in [0, 1], i = 1, \dots, 6 \right\}$$

is a zonotope, where $p^1 = \mu([0, 1)) = (30, 40, 10)$, $p^2 = \mu([1, 2)) = (40, 10, 20)$, $p^3 = \mu([2, 3)) = (10, 20, 20)$, $p^4 = \mu([3, 4)) = (10, 20, 30)$, $p^5 = \mu([4, 5)) = (15, 10, 20)$, $p^6 = \mu([5, 6)) = (5, 30, 25)$.

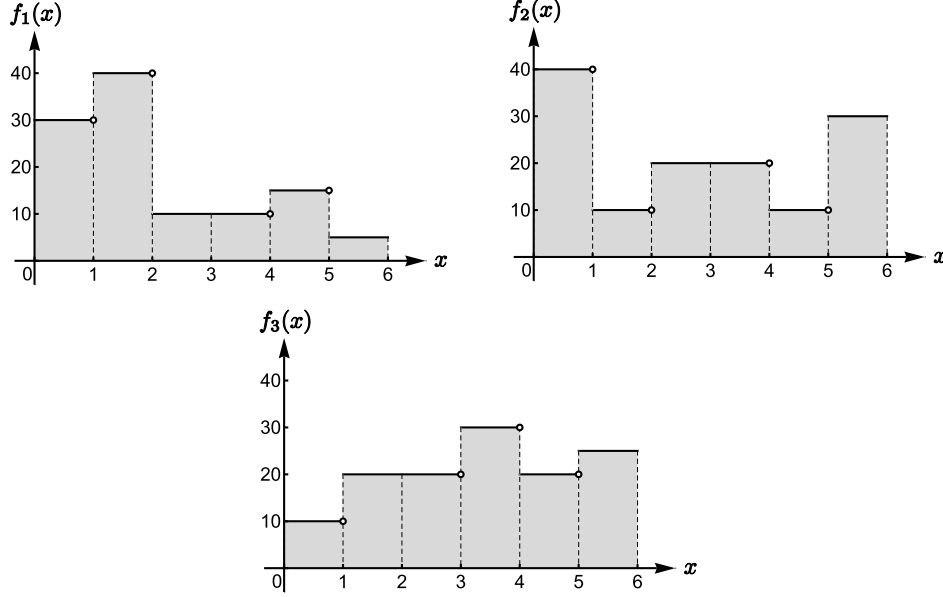


FIGURE 1. Density functions of the vector measure in Example 3.1.

Let $p = p^1 + p^2 + p^3 = (80, 70, 50)$. Observe that p is an extreme point of $R_\mu(X)$. Indeed, consider the vector $d = (\frac{7}{5}, 1, -\frac{8}{5})$ and the linear function $l_d(\alpha)$ defined for all $\alpha \in \mathbb{R}^6$ by the scalar product

$$\begin{aligned} l_d(\alpha) &= d \cdot \left(\sum_{i=1}^6 \alpha_i p^i \right) = \sum_{i=1}^6 \alpha_i (d \cdot p^i) \\ &= 66\alpha_1 + 34\alpha_2 + 2\alpha_3 - 14\alpha_4 - \alpha_5 - 3\alpha_6. \end{aligned}$$

On the set $R_\mu(X)$, explicitly presented in (3.1), this function achieves a unique maximum at the point $\alpha^* = (1, 1, 1, 0, 0, 0)$ with $l_d(\alpha^*) = 66 + 34 + 2 = 102$. In addition, $\sum_{i=1}^6 \alpha_i^* p^i = p$. So, $d \cdot r - 102 \leq 0$ for all $r \in R_\mu(X)$ and the equality holds if and only if $r = p$. Thus, $d \cdot r - 102 = 0$ is a supporting hyperplane of the convex polytope $R_\mu(X)$, and the intersection of the polytope and hyperplane consists of the single point p . This implies that p is an extreme point of $R_\mu(X)$.

According to the definition of $R_\mu(X)$, for $p \in R_\mu(X)$ there exists a measurable subset $Z \in \mathcal{F}$ such that $\mu(Z) = p$ and, according to [9, Theorem III] described in Section 1, since p is extreme, such Z is unique up to null sets. In particular, $p = \mu(Z)$ for $Z = [0, 3]$. Thus,

$$R_\mu^p(X) = R_\mu(Z) = \left\{ \sum_{i=1}^3 \alpha_i p^i : \alpha_i \in [0, 1], i = 1, 2, 3 \right\}.$$

Choose $q = (56, 29, 31)$ and observe that $q \notin R_\mu^p(X)$. Indeed, $q \in R_\mu^p(X)$ if and only if there exist $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$, such that $\sum_{i=1}^3 \alpha_i p^i = q$, which is equivalent to

$$(3.2) \quad \alpha_1(30, 40, 10) + \alpha_2(40, 10, 20) + \alpha_3(10, 20, 20) = (56, 29, 31),$$

but the only solution to the linear system of equations (3.2) is

$$\alpha_1 = \frac{3}{10}, \alpha_2 = \frac{11}{10}, \alpha_3 = \frac{3}{10},$$

where $\alpha_2 \notin [0, 1]$.

On the other hand, $q \in Q_\mu^p(X)$, because: (i) $q \in R_\mu(X)$, and (ii) $q \in R_\mu(X) - \{\mu(X) - p\}$. Indeed, (i) holds since, for $Z_1 = [0, \frac{42}{115}) \cup [1, 1\frac{229}{230}) \cup [2, 2\frac{33}{460}) \cup [4, 4\frac{3}{10})$,

$$\begin{aligned} \mu(Z_1) &= \frac{42}{115} \times (30, 40, 10) + \frac{229}{230} \times (40, 10, 20) \\ &+ \frac{33}{460} \times (10, 20, 20) + \frac{3}{10} \times (15, 10, 20) \\ &= (56, 29, 31) = q. \end{aligned}$$

Notice that (ii) is equivalent to $q + \mu(X) - p \in R_\mu(X)$, where $q + \mu(X) - p = (56, 29, 31) + (110, 130, 125) - (80, 70, 50) = (86, 89, 106)$. Let $Z_2 = [0, \frac{15}{46}) \cup [1, 1\frac{45}{46}) \cup [2, 2\frac{209}{230}) \cup [3, 5) \cup [5, 5\frac{3}{5})$. Then

$$\begin{aligned} \mu(Z_2) &= \frac{15}{46} \times (30, 40, 10) + \frac{45}{46} \times (40, 10, 20) + \frac{209}{230} \times (10, 20, 20) \\ &+ 1 \times (10, 20, 30) + 1 \times (15, 10, 20) + \frac{3}{5} \times (5, 30, 25) \\ &= (86, 89, 106) = q + \mu(X) - p. \end{aligned}$$

Thus (ii) holds too, and $R_\mu^p(X) \neq Q_\mu^p(X)$. \square

The following example demonstrates that the necessary condition (1.6) for the existence of a measurable partition $\{X^a : a \in A\}$ with $\mu(X^a) = p^a$, $a \in A$, is not sufficient for an atomless measure μ when $m > 2$. In this example, A consists of three points. According to [1, Theorem 2.5], this condition is necessary and sufficient when $m = 2$, A is countable, and μ is atomless. If A consists of two points, say a and b , and $p^a \in R_\mu(X)$, $p^b = \mu(X) - p^a$, then the partition $\{X^a, X^b\}$ always exists with X^a selected as any $X^a \in \mathcal{F}$ satisfying $\mu(X^a) = p^a$ and with $X^b = X \setminus X^a$.

Example 3.2. Consider the measure space (X, \mathcal{B}, μ) defined in Example 3.1. Let $p^1 = (56, 29, 31)$, $p^2 = (24, 41, 19)$, $p^3 = (30, 60, 75)$, and $A = \{1, 2, 3\}$. Then $p^1 + p^2 + p^3 = \mu(X)$. We further observe that: (i) p^1 is the vector q from Example 3.1, so $p^1 \in R_\mu(X)$ and therefore $p^2 + p^3 = \mu(X) - p^1 \in R_\mu(X)$; (ii) $p^1 + p^3$ is the vector $q + \mu(X) - p$ from Example 3.1, so $p^1 + p^3 \in R_\mu(X)$ and therefore $p^2 = \mu(X) - p^1 - p^3 \in R_\mu(X)$; (iii) $p^1 + p^2$ is the vector p from Example 3.1, so $p^1 + p^2 \in R_\mu(X)$ and therefore $p^3 = \mu(X) - p^1 - p^2 \in R_\mu(X)$. Thus, the vectors p^a , $a \in A$, satisfy (1.6).

If there exists a partition $\{X^a \in \mathcal{B} : a \in A\}$ of X with $\mu(X^a) = p^a$ for all $a \in A$, let $Y = X^1 \cup X^2$. Since $X^1 \cap X^2 = \emptyset$, $\mu(X^1) = p^1 = q$, and $\mu(Y) = p^1 + p^2 = p$, then $q \in R_\mu^p(X)$. According to Example 3.1, $q \notin R_\mu^p(X)$. This contradiction implies that a partition $\{X^a \in \mathcal{B} : a \in A\}$ of X , with $\mu(X^a) = p^a$ for all $a \in A$, does not exist. \square

In conclusion, we provide two simple examples showing that, if μ is not atomless, then even for $m = 1$ (and, therefore, for any natural number m) a maximal subset (1.1) may not exist and equation (1.2) may not hold.

Example 3.3. Consider the measure space $(X, 2^X, \mu)$, where $X = \{1, 2, 3, 4\}$ and

$$\mu(\{1\}) = 0.1, \mu(\{2\}) = 0.4, \mu(\{3\}) = 0.2, \mu(\{4\}) = 0.3.$$

Let $p = 0.5$. Then $\mathcal{S}_\mu^p = \{\{1, 2\}, \{3, 4\}\}$. In other words, the only subsets that have the measure 0.5 are $Z^1 = \{1, 2\}$ and $Z^2 = \{3, 4\}$. However, $R_\mu(Z^1)$ is not a subset of $R_\mu(Z^2)$ and vice versa. Therefore, a maximal subset does not exist for $p = 0.5$. \square

Example 3.4. Consider the measure space $(X, 2^X, \mu)$, where $X = \{1, 2, 3\}$ and

$$\mu(\{1\}) = 0.1, \mu(\{2\}) = 0.55, \mu(\{3\}) = 0.35.$$

The range of μ on X is $R_\mu(X) = \{0, 0.1, 0.35, 0.45, 0.55, 0.65, 0.9, 1\}$. Let $p = 0.55$. Then $Q_\mu^p = \{0, 0.1, 0.45, 0.55\}$ and $R_\mu^p = \{0.55\}$. Thus $R_\mu^p \subset Q_\mu^p$, but $R_\mu^p \neq Q_\mu^p$. \square

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DEPARTMENT OF APPLIED MATHEMATICS AND STATISTICS, STONY BROOK UNIVERSITY, STONY BROOK, NY 11794-3600

E-mail address: Peng.Dai@stonybrook.edu

DEPARTMENT OF APPLIED MATHEMATICS AND STATISTICS, STONY BROOK UNIVERSITY, STONY BROOK, NY 11794-3600

E-mail address: Eugene.Feinberg@sunysb.edu